IOP Publishing Journal of Physics A: Mathematical and Theoretical

J. Phys. A: Math. Theor. **49** (2016) 055203 (21pp) doi:10.1088/1751-8113/49/5/055203

Stability of coefficients in the Kronecker product of a hook and a rectangle

Cristina M Ballantine[1,3] **and William T Hallahan**[2,4]

[1] College of the Holy Cross, USA
[2] College of the Holy Cross and Yale University, USA

E-mail: cballant@holycross.edu and william.hallahan@yale.edu

Received 10 August 2015, revised 27 November 2015
Accepted for publication 3 December 2015
Published 30 December 2015

Abstract

We use recent work of Jonah Blasiak (2012 arXiv:1209.2018) to prove a stability result for the coefficients in the Kronecker product of two Schur functions: one indexed by a hook partition and one indexed by a rectangle partition. We also give nearly sharp bounds for the size of the partition starting with which the Kronecker coefficients are stable. Moreover, we show that once the bound is reached, no new Schur functions appear in the decomposition of Kronecker product. We call this property *superstability*. Thus, one can recover the Schur decomposition of the Kronecker product from the smallest case in which the superstability holds. The bound for superstability is sharp. Our study of this particular case of the Kronecker product is motivated by its usefulness for the understanding of the quantum Hall effect (Scharf T *et al* 1994 *J. Phys. A: Math. Gen* **27** 4211–9).

Keywords: Schur functions, Kronecker product, stability, q-discriminant, quantum Hall effect

1. Introduction

Let χ^λ and χ^μ be the irreducible characters of the symmetric group on n letters, S_n, indexed by the partitions λ and μ of n. The *Kronecker product* $\chi^\lambda\chi^\mu$ is defined by $(\chi^\lambda\chi^\mu)(w) = \chi^\lambda(w)\chi^\mu(w)$ for all $w \in S_n$. Thus, $\chi^\lambda\chi^\mu$ is the character that corresponds to the diagonal action of S_n on the tensor product of the irreducible representations indexed by λ and μ. Then, we have

[3] This work was partially supported by a grant from the Simons Foundation (#245997 to Cristina Ballantine).
[4] Author to whom any correspondence should be addressed.

$$\chi^{\lambda}\chi^{\mu} = \sum_{\nu \vdash n} g(\lambda, \mu, \nu)\chi^{\nu},$$

where $g(\lambda, \mu, \nu)$ is the multiplicity of χ^{ν} in $\chi^{\lambda}\chi^{\mu}$. Hence, the numbers $g(\lambda, \mu, \nu)$ are non-negative integers.

By means of the Frobenius map, one can define the Kronecker (internal) product on the Schur symmetric functions by

$$s_{\lambda} * s_{\mu} = \sum_{\nu \vdash n} g(\lambda, \mu, \nu) s_{\nu}.$$

A reasonable formula for decomposing the Kronecker product is unavailable, although the problem has been studied since the early twentieth century. Some results exist in particular cases. Lascoux [LA], Remmel [R], Remmel and Whitehead [RW], and Rosas [Ro] derived closed formulas for Kronecker products of Schur functions indexed by two row partitions or hook partitions. Dvir [D], and Clausen and Meier [CM] have given, for any λ and μ, a simple and precise description for the maximum length of ν and the maximum size of ν_1 whenever $g(\lambda, \mu, \nu)$ is non-zero. Bessenrodt and Kleshchev [BK] have looked at the problem of determining when the decomposition of the Kronecker product has one or two constituents. Similarly, combinatorial interpretations of the Kronecker coefficients $g(\lambda, \mu, \nu)$ exist only in particular cases: (i) if λ and μ are both hooks [R]; (ii) if λ is a two row partition (with some conditions on the size of the first part) [BO-2]; (iii) if λ is a hook partition [B]; (iv) if λ and μ are both two row partitions [BMS]. Recent years have seen a resurgence of the study of the Kronecker product, motivated by application to geometric complexity theory and quantum information theory. In the first seminal paper on the Kronecker product, Murnaghan [M] observed the following stability property. Given the partitions $\bar{\lambda}$, $\bar{\mu}$, $\bar{\nu}$ of a, b, c respectively, define $\lambda(n) := (n - a, \bar{\lambda})$, $\mu(n) := (n - b, \bar{\mu})$, and $\nu(n) := (n - c, \bar{\nu})$. Then, the Kronecker coefficient $g(\lambda(n), \mu(n), \nu(n))$ does not depend on n for n larger than some integer $N = N(\bar{\lambda}, \bar{\mu}, \bar{\nu})$. We say that the Kronecker coefficients $g((n - a, \bar{\lambda}), (n - b, \bar{\mu}), (n - c, \bar{\nu}))$ *stabilize* for $n \geqslant N$. Proofs of this stability property and lower bounds for N were given by Brion [Br], using algebraic geometry, and Vallejo, [V1, V2] using combinatorics of the Young tableaux. More generalized stability notions make sense. These were first observed by Manivel [Man]. Generalized stability has also been recently studied by Stembridge [St]. He refers to a triple of partitions (α, β, γ) as *stable* if, for any other triple of partitions (λ, μ, ν), the Kronecker coefficient $g(n\alpha + \lambda, n\beta + \mu, n\gamma + \nu)$ does not depend on n for n large enough. Stembridge's conjecture was proved by Sam and Snowden [SS]. Pak and Panova [PP], and independently Vallejo [V3], recently gave alternate proofs of k-stability for Kronecker coefficients, i.e., stability for the triple $(1^k, k, 1^k)$.

In this article, using Blasiak's work [B], we investigate the stability of the Kronecker coefficients $g(\lambda, \mu, \nu)$ when $\lambda = (m^t)$ is a rectangle partition of $n = mt$ and $\mu = (n - d, 1^d)$ is a hook partition of n. Specifically, if $\nu^{(m)}$ is the partition obtained from ν by adding a row of length m and reordering the parts to form a partition, then whenever $t \geqslant d + 1$ we have

$$g\Big(\big(m^t\big), \big(n - d, 1^d\big), \nu\Big) = g\Big(\big(m^{t+1}\big), \big(n - d + m, 1^d\big), \nu^{(m)}\Big).$$

We note that, using conjugates of partitions, this result is a special case of k-stability. However, our proof is completely different from all previous proofs in the literature (see [PP, St, V3]). Since the stability property fails if $t = d - 1$, our method gives a nearly sharp bound for t, starting with which stability holds. Moreover, this bound depends only on t and d, while the bound in [V3] depends on all three partitions. In fact, our method allows us to prove a much stronger stability property, as explained below.

Figure 1. $\lambda = (6, 4, 2, 1, 1)$, $\ell(\lambda) = 5$, $|\lambda| = 14$ and $\lambda' = (5, 3, 2, 2, 1, 1)$

If $t \geqslant d + 1$, all Schur functions appearing in the decomposition of $s_{(n-d+m,1^d)} * s_{(m^{t+1})}$ are of the form $s_{\nu^{(m)}}$ for a partition ν, such that s_ν appears in the decomposition of $s_{(n-d,1^d)} * s_{(m^t)}$. Thus, if $n = m(d + 1)$, one can completely recover the decomposition of the Kronecker product $s_{(n-d+km,1^d)} * s_{(m^{d+1+k})}$ from the decomposition of Kronecker product $s_{(n-d,1^d)} * s_{(m^{d+1})}$. We call this property *superstability* of Kronecker coefficients. Superstability fails if $t = d$. Thus, our bound, $t = d + 1$, starting with which superstability holds, is sharp.

Our study of the particular case of the Kronecker product of a hook partition and a rectangular partition is motivated by its usefulness for the understanding of the quantum Hall effect [STW]. The connection between the fractional quantum Hall effect and the decomposition of even powers of the Vandermonde determinant in the Schur basis has been described in many articles. We refer the reader to the introduction of [STW] and the references therein. In [STW], the authors give two algorithms for computing the coefficients in the decomposition. They note that 'the main inconvenience of these methods is that they give only the complete expansion. To investigate the individual coefficients, one probably has to look at the q-analogue. In this case, the coefficients have a nice expression, although they are not amenable to practical computations.' In fact, the authors show that the problem of finding the coefficients in the decomposition of the q-discriminant is equivalent to finding the Kronecker product of a square character and a hook character. Since we are now interested in finding the coefficients in the decomposition of $s_{(m^m)} * s_{(m^2-d,1^d)}$, if $d < m$, we can reduce the problem to finding the coefficients in $s_{(m^{d+1})} * s_{(m(d+1)-d,1^d)}$. In fact, using the S_3 symmetry of the Kronecker coefficients and the fact the $s_\lambda * s_\mu = s_{\lambda'} * s_{\mu'}$, where λ' is the conjugate of λ, one can reduce the calculation even further to the product $s_{((d+1)^{d+1})} * s_{(d^2+d+1,1^d)}$. Of course, this reduction of the problem to a much smaller dimension only helps if the leg (or the arm) of the hook is shorter than the side of the square.

2. Preliminaries and notation

In this section we set the notation and introduce some basic background about partitions and Schur functions, mostly following [BO-1]. For details and proofs of the contents of this section see [Ma] or [S]. Let n be a non-negative integer. A *partition* of n is a weakly decreasing sequence of non-negative integers, $\lambda \coloneqq (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$, such that $|\lambda| = \sum \lambda_i = n$. We write $\lambda \vdash n$ to mean λ is a partition of n. The non-zero integers λ_i are called the *parts* of λ. We identify a partition with its *Young diagram*, i.e. the array of left-justified squares (boxes) with λ_1 boxes in the first row, λ_2 boxes in the second row, and so on. The rows are arranged in matrix form from top to bottom. The *size* of λ is $|\lambda| = \sum \lambda_i$. The *length* of λ, $\ell(\lambda)$, is the number of rows in the Young diagram. Given a partition λ, its *conjugate* is the partition λ' whose Young diagram's rows are precisely the columns of λ. For an example, see figure 1.

Given two partitions λ and μ, we write $\mu \subseteq \lambda$ if and only if $\ell(\mu) \leqslant \ell(\lambda)$ and $\lambda_i \geqslant \mu_i$ for $1 \leqslant i \leqslant \ell(\mu)$. If $\mu \subseteq \lambda$, we denote by λ/μ the skew shape obtained by removing the boxes corresponding to μ from λ. For an example, see figure 2.

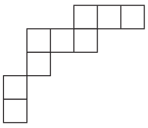

Figure 2. λ/μ where $\lambda = (6, 4, 2, 1, 1)$ and $\mu = (3, 1, 1)$.

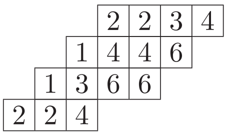

Figure 3. A SSYT of shape $\lambda/\mu = (7, 6, 5, 3)/(3, 2, 1)$ and type $(2, 4, 2, 4, 0, 3)$.

A *semi-standard Young tableau* (SSYT) *of shape* λ/μ is a filling of the boxes of the skew shape λ/μ with positive integers so that the numbers weakly increase in each row from left to right and strictly increase in each column from top to bottom. The *type* of a SSYT T is the sequence of non-negative integers $(t_1, t_2, \ldots)$, where t_i is the number of is in T. We write $T_{i,j}$ to refer to the filling of the box in the ith row and jth column of T.

If T is a SSYT of shape λ/μ and type $(t_1, t_2, \ldots)$, we define its *weight*, $w(T)$, to be the monomial obtained by replacing each i in T by x_i and taking the product over all boxes, i.e. $w(T) = x_1^{t_1} x_2^{t_2} \ldots$. For example, the weight of the SSYT in figure 3 is $x_1^2 x_2^4 x_3^2 x_4^4 x_6^3$. The skew Schur function $s_{\lambda/\mu}$ is defined combinatorially by the formal power series

$$s_{\lambda/\mu} = \sum_T w(T),$$

where the sum runs over all SSYTs of shape λ/μ. To obtain the usual Schur function one sets $\mu = \varnothing$.

The space of homogeneous symmetric functions of degree n is denoted by Λ^n. A basis for this space is given by the Schur functions $\{s_\lambda \mid \lambda \vdash n\}$. The Hall inner product on Λ^n is denoted by $\langle , \rangle_{\Lambda^n}$ and it is defined by

$$\left\langle s_\lambda, s_\mu \right\rangle_{\Lambda^n} = \delta_{\lambda\mu},$$

where $\delta_{\lambda\mu}$ denotes the Kronecker delta.

For a positive integer r, let $p_r = x_1^r + x_2^r + \cdots$. Then $p_\mu = p_{\mu_1} p_{\mu_2} \ldots p_{\mu_{\ell(\mu)}}$ is the power symmetric function corresponding to the partition μ of n. If CS_n denotes the space of class functions of S_n, then the *Frobenius characteristic map* F: $CS_n \to \Lambda^n$ is defined by

$$F(\sigma) = \sum_{\mu \vdash n} z_\mu^{-1} \sigma(\mu) p_\mu,$$

where $z_\mu = 1^{m_1} m_1!\, 2^{m_2} m_2! \ldots n^{m_n} m_n!$ if $\mu = (1^{m_1}, 2^{m_2}, \ldots, n^{m_n})$, i.e. k is repeated m_k times in μ, and $\sigma(\mu) = \sigma(\omega)$ for an $\omega \in S_n$ of cycle type μ. Note that F is an isometry. If χ^λ is an irreducible character of S_n then, by the Murnaghan–Nakayama rule [S], $F(\chi^\lambda) = s_\lambda$.

We define the *Kronecker product* of Schur functions by

$$s_\lambda * s_\mu = \sum_{\nu \vdash n} g(\lambda, \mu, \nu) s_\nu,$$

where $g(\lambda, \mu, \nu)$ is the multiplicity of χ^ν in $\chi^\lambda \chi^\mu$.

A *word* is a sequence of letters from some totally ordered set called an *alphabet*. A word $a_1a_2\ldots a_n$ in the alphabet $\{1, 2, \ldots\}$ is called a *lattice permutation* (or *Yamanouchi*) if in any initial factor $a_1a_2\cdots a_j$, the number of i's is at least as great as the number of $(i+1)$'s for all i. For example, 11 122 321 is a lattice permutation. The *reverse reading word* (rrw) of a tableau is the sequence of entries of T obtained by reading the entries from right to left and top to bottom. For example, the rrw of the tableau in figure 3 is 432 264 416 631 422.

3. Blasiak's combinatorial rule

In this section, following [B], we give a brief description of the combinatorial interpretation of the Kronecker coefficients when one of the Schur functions is indexed by a hook partition. All partitions in this section are of size n. We write $\mu(d)$ for the hook partition $(n-d, 1^d)$. First, we introduce the necessary notation.

The set $\{1, 2, \ldots\}$ is called the *alphabet of unbarred (or ordinary) letters*, and the set $\{\bar{1}, \bar{2}, \ldots\}$ is called the *alphabet of barred letters*. A *colored word* is a word in the alphabet $\mathcal{A} = \{1, 2, \ldots\} \cup \{\bar{1}, \bar{2}, \ldots\}$. We will need the following orders on $\mathcal{A}$.

the natural order: $\bar{1} < 1 < \bar{2} < 2 < \cdots$
the small bar order: $\bar{1} \prec \bar{2} \prec \cdots \prec 1 \prec 2 \prec \cdots$

$<^k$*order*:$\bar{1} <^k \bar{2} <^k \cdots <^k \bar{k} <^k 1 <^k 2 <^k \cdots$
$<^k k <^k \overline{k+1} <^k k+1 <^k \overline{k+2} <^k k+2 <^k \cdots$.

Clearly, $<^1 = <$ and $<^\infty = \prec$.

A *semistandard colored tableau*, for any of the above orders on $\mathcal{A}$, is a tableau with entries in $\mathcal{A}$ such that: (i) unbarred letters increase weakly from left to right in each row and strictly from top to bottom in each column; (ii) barred letters increase strictly from left to right in each row and weakly from top to bottom in each column.

Given a semistandard colored tableau T for the order $<^k$, it can be converted to a tableau for order $<^{k-1}$ by performing exchanges between each $\bar{k}$ and the letters $\{1\ldots k\}$ as follows. Let β be the bottommost $\bar{k}$. If β is greater than it's neighbor below or to its right in the $<^{k-1}$ order, swap with the lesser (or only) neighbor. Favor the neighbor below, if the neighbors are equal. Repeat until β can no longer be exchanged with any neighbor. Repeat the process with the second bottommost $\bar{k}$, and so on. Converting from the order $<^{k-1}$ to the order $<^k$ is obtained by simply reversing the steps.

Given a semistandard colored tableau for any one of the above orders, one can convert it to a semistandard colored tableau for another order using Jeu-de-Taquin moves as described above. Conversion from $\prec$ to $<$ (or $<$ to $\prec$) may be regarded as repeated conversion from $<^k$ to $<^{k-1}$ (respectively, $<^k$ to $<^{k+1}$). In figure 4, we use example 2.17 in [B] to show the conversion of a semistandard colored tableau for the small bar order to a semistandard colored tableau for the natural bar order. Converting from the natural order to the small bar ordered is obtained by simply reversing the steps.

The *content* of a colored tableau T is $c = (c_1, c_2, \ldots)$, where c_i is the number of i and $\bar{i}$ in T. The *total color* of T is the number of barred letters in T.

Let $T^<$ be a colored tableau for natural order $<$ and let $T^\prec$ be the tableau obtained from $T^<$ by converting to the small bar order $\prec$. Let T^b be the tableau of barred letters in $T^\prec$. Denote by T^{sk} the tableau obtained by placing $T^\prec/T^b$ above and to the right of $(T^b)'$, so that the NE corner of $(T^b)'$ touches the SW corner of $T^\prec/T^b$, and removing the bars from the letters of $(T^b)'$.

$$
\begin{array}{cccc} \bar{1} & \bar{2} & \bar{3} & 1 \\ \bar{1} & \bar{3} & \bar{4} & 2 \\ \bar{2} & 1 & 1 & 3 \\ 1 & 2 & 4 & \\ 3 & 5 & & \end{array}
\leftrightarrow
\begin{array}{cccc} \bar{1} & \bar{2} & \bar{3} & 1 \\ \bar{1} & \bar{3} & 1 & 2 \\ \bar{2} & 1 & 3 & \bar{4} \\ 1 & 2 & 4 & \\ 3 & 5 & & \end{array}
\leftrightarrow
\begin{array}{cccc} \bar{1} & \bar{2} & \bar{3} & 1 \\ \bar{1} & 1 & 1 & 2 \\ \bar{2} & 2 & 3 & \bar{4} \\ 1 & \bar{3} & 4 & \\ 3 & 5 & & \end{array}
\leftrightarrow
\begin{array}{cccc} \bar{1} & \bar{2} & 1 & 1 \\ \bar{1} & 1 & 2 & \bar{3} \\ \bar{2} & 2 & 3 & \bar{4} \\ 1 & \bar{3} & 4 & \\ 3 & 5 & & \end{array}
\leftrightarrow
\begin{array}{cccc} \bar{1} & \bar{2} & 1 & 1 \\ \bar{1} & 1 & 2 & \bar{3} \\ 1 & 2 & 3 & \bar{4} \\ \bar{2} & \bar{3} & 4 & \\ 3 & 5 & & \end{array}
\leftrightarrow
\begin{array}{cccc} \bar{1} & 1 & 1 & 1 \\ \bar{1} & \bar{2} & 2 & \bar{3} \\ 1 & 2 & 3 & \bar{4} \\ \bar{2} & \bar{3} & 4 & \\ 3 & 5 & & \end{array}
$$

Figure 4. Blasiak's example 2.17.

In figure 5, we show an example of the tableaux $T^{<}$, $T^{\prec}$, T^{b}, and T^{sk}, where $T^{<}$ is the tableau on the right in figure 4.

We refer to a semistandard Young tableau (for the natural order $<$ or by means of the Frobenius map one can define the Kronecker (internal) product on the Schur symmetric functions by bar order $\prec$) as a *colored tableau*. A colored tableau T is called *Yamanouchi* if the rrw of T^{sk} is a lattice permutation.

A colored tableau $T^{<}$ is *color raisable* if the box in the SW corner has an unbarred letter. The notion of *color raisable* makes sense only for tableaux using the natural order.

In the rule below, the colored tableaux are in the natural order.

Blasiak's combinatorial rule: The Kronecker coefficient $g(\lambda, \mu(d), \nu)$ equals the number of color raisable Yamanouchi colored tableaux of shape ν, content λ, and total color d.

Given a colored word w, the the *unbarred word* of w, denoted $w_{\varnothing}$, is the word obtained from w by removing all barred letters.

We end this section by showing that, if we start with a Yamanouchi colored tableau in small bar order and convert to the natural order, after every Jeu-de-Taquin move the unbarred rrw of the new tableau is a lattice permutation.

Proposition 3.1. *Let $T^{\prec}$ be a colored Yamanouchi tableau. The unbarred rrw of the tableau obtained by converting $T^{\prec}$ to the $<^{k}$ order (for some $k > 0$) is a lattice permutation.*

Proof. Let T be a tableau in some intermediate state between $T^{\prec}$ and its conversion to the order $<^{k}$, and let T' be the tableau obtained after the next Jeu-de-Taquin move in T. Let w, respectively w', be the unbarred rrw of T, respectively T'. We will show that if $w_{\varnothing}$ is Yamanouchi, then so is $w'_{\varnothing}$.

Let $\bar{x}$ be the letter moved to obtain T' from T. We use the following notation in T: r is the letter directly to the right of $\bar{x}$ (if it exists), and q is the letter directly below $\bar{x}$ (if it exists). Note that r and q could be barred or unbarred. If it exists, we denote by p the letter that was bumped by $\bar{x}$ when it arrived in its current position in T. Note that p is necessarily unbarred. We refer to the row of $\bar{x}$ in T as row B. Let u be the number of $q - 1$ to the left of q in $w_{\varnothing}$.

If, to get from T to T', $\bar{x}$ shifts to the right, then $w_{\varnothing} = w'_{\varnothing}$. Now suppose $\bar{x}$ moves down to row $B + 1$ (bumping q up to row B). Then q is unbarred, and either $q \leqslant^{k} r$ or r does not exist. Moreover, in row B, there is no $q - 1$ to the right of r. We consider three cases.

(i) p does not exist, i.e. this is the first time $\bar{x}$ is moving. In T, all the letters in B to the left of $\bar{x}$ are barred, and there is no $q - 1$ in row B. Then, all u letters $q - 1$ must be above row B, and moving q up one row results in $w'_{\varnothing}$ Yamanouchi.

(ii) $p = q - 1$. If the last move of $p = q - 1$ (before obtaining T) was up, clearly moving q up will result in $w'_{\varnothing}$ Yamanouchi. Now consider the case when the last move of $p = q - 1$ was left. If, at some point prior to this move, $\bar{x}$ moved down to row B by

$$
\begin{array}{|c|c|c|c|}\hline \bar{1} & 1 & 1 & 1 \\ \hline \bar{1} & \bar{2} & 2 & \bar{3} \\ \hline 1 & 2 & 3 & \bar{4} \\ \hline \bar{2} & \bar{3} & 4 \\ \cline{1-3} 3 & 5 \\ \cline{1-2} \end{array}
\qquad
\begin{array}{|c|c|c|c|}\hline \bar{1} & \bar{2} & \bar{3} & 1 \\ \hline \bar{1} & \bar{3} & \bar{4} & 2 \\ \hline \bar{2} & 1 & 1 & 3 \\ \hline 1 & 2 & 4 \\ \cline{1-3} 3 & 5 \\ \cline{1-2} \end{array}
\qquad
\begin{array}{|c|c|c|}\hline \bar{1} & \bar{2} & \bar{3} \\ \hline \bar{1} & \bar{3} & \bar{4} \\ \hline \bar{2} \\ \cline{1-1} \end{array}
\qquad
\begin{array}{ccccccc}
 & & & & & & \boxed{1} \\
 & & & & & & \boxed{2} \\
 & & & & \boxed{1} & \boxed{1} & \boxed{3} \\
 & & & \boxed{1} & \boxed{2} & \boxed{4} & \\
 & & & \boxed{3} & \boxed{5} & & \\
 \boxed{1} & \boxed{1} & \boxed{2} & & & & \\
 \boxed{2} & \boxed{3} & & & & & \\
 \boxed{3} & \boxed{4} & & & & &
\end{array}
$$

$T^{<}$ $T^{\prec}$ T^{b} T^{sk}

Figure 5. An example of tableaux $T^{<}$, $T^{\prec}$, T^{b}, T^{sk}.

bumping a letter $q - 1$, then $w'_{\emptyset}$ is Yamanouchi. If $\bar{x}$ was always in row B or moved down to row B by bumping a letter less than $q - 1$, then the number of q in row $B + 1$ is at least one more than the number of $q - 1$ in row B (when $\bar{x}$ arrives in row B directly to the left of the first $q - 1$, each $q - 1$ has a q immediately under it, and there must be a q directly under $\bar{x}$). Then, there is at least one extra $q - 1$ in T above row B. Thus, moving q up results in $w'_{\emptyset}$ Yamanouchi.

(iii) $p < q - 1$. Since $q \leqslant^{k} r$, or r does not exist, there are no $q - 1$'s in row B. As in (i), $w'_{\emptyset}$ is Yamanouchi. □

4. On the Kronecker product of a Hook and a rectangle

In this section, we collect several properties of the partitions indexing Schur functions that appear in the Kronecker product of the Schur function associated with the hook partition $(n - d, 1^{d})$ and the Schur function associated with the rectangular partition (m^{t}), with $mt = n$. For the remainder of the article, $T^{<}$ will denote a semistandard Young tableau for the natural order and $T^{\prec}$, T^{b}, and T^{sk} will denote the tableaux associated with $T^{<}$, defined in the previous section. In addition, η will always denote the shape of T^{b}, and all colored tableau will have total color d. Given a colored Yamanouchi tableau $T^{\prec}$ (or $T^{<}$) of shape ν, content (m^{t}), and total color d, we denote by T_1 the tableau formed by the first $\ell(\eta)$ rows of $T^{\prec}/T^{b}$ and by T_2 the tableau formed by the remaining rows of $T^{\prec}/T^{b}$. We write $\xi = (\xi_1, \xi_2, \ldots, \xi_{\ell(\eta)})$ for the content of T_1 and $\zeta = (\zeta_1, \zeta_2, \ldots, \zeta_{\ell(\eta)+1})$ for the content of row $\ell(\eta) + 1$ of $T^{\prec}$.

Recall that our goal is to show that if $t = d + w$, $w \geqslant 1$, then

$$
g\Big(\big(m^{t}\big), \big(n - d, 1^{d}\big), \nu\Big) = g\Big(\big(m^{t+1}\big), \big(n - d + m, 1^{d}\big), \nu^{(m)}\Big), \tag{1}
$$

where $\nu^{(m)}$ is obtained from ν by adding a part of size m. Moreover, we will show that every partition indexing a Schur function appearing in $s_{(m^{t+1})} * s_{(n-d+m, 1^{d})}$ is obtained from a partition indexing a Schur function appearing in $s_{(m^{t})} * s_{(n-d, 1^{d})}$ by adding a part of size m. To prove this, we will define a bijection of color raisable Yamanouci tableaux, defined by inserting a row of length m after row $\ell(\eta)$ and changing the filling of the tableau according to certain rules. Towards our goal, we need to understand some properties of possible shapes of tableaux occurring, as well as the filling of certain rows of these tableaux. We note that if $m = 1$, we have $s_{(1^{t})} * s_{(t-d, 1^{d})} = s_{(d+1, 1^{t-d-1})}$. Similarly, if $d = 0$, we have $s_{(m^{t})} * s_{(mt)} = s_{(m^{t})}$. In both cases the superstability property holds if $t = d + w$ with $w \geqslant 1$. Thus, for the remainder of the paper we can assume that $m \geqslant 2$ and $d \geqslant 1$.

In the next theorem, we prove an important property of tableaux counted by Blasiak's rule to determine $g((m^t), (n - d, 1^d), \nu)$. Here, the fact that the tableaux are color raisable is irrelevant.

Theorem 4.1. *Given a colored Yamanouchi tableau $T^<$ of shape ν, content (m^t), and total color* d*, the shape of T^b completely determines the tableau T^b.*

Proof. By the definition of a semistandard colored tableau, T^{sk} must be an ordinary SSYT of type (m^t). Moreover, its rrw is a lattice permutation. Let η be the shape of T^b. In T^{sk}, the shape of $(T^b)'$ is η'. Since there are exactly m of each letter appearing in T^{sk}, the rrw of T^{sk} must end in t, the highest available letter, and therefore the last row of T^{sk}, which has length η'_{η_1}, is filled with t. Since the content of T^{sk} is a rectangle, the second to last row in T^{sk} must start with $t - 1$ and thus can only be filled with the letters $t - 1$ and t. By the semistandard condition, the first η'_{η_1} entries in this row must be $t - 1$. Moreover, by the lattice permutation condition and the fact that the content is a rectangle, there cannot be more than η'_{η_1} letters $t - 1$ in the second to last row in T^{sk}. Continuing in this way, we see that the content of T^b is determined. In fact, each row of T^b is filled in decreasing order from right to left with the letters $\bar{t}, \overline{t-1}, \overline{t-2}, \ldots$. $\square$

The next observation is an immediate consequence of the proof of theorem 4.1.

Remark 4.2. Let $T^<$ be a colored Yamanouchi tableau of shape ν, content (m^t), and total color d. The barred letters in T^b are precisely $\bar{t}, \overline{t-1}, \ldots, \overline{t - \eta_1 + 1}$. Moreover, the partition obtained by reading the parts of the content of T^b (i.e., of only the barred letters) in reverse order is precisely η'.

Next, we highlight two properties of colored Yamanouchi tableaux of shape ν, content (m^t), and total color d, in the case when $t = d + w$, where $w \geqslant 1$.

Proposition 4.3. *Let $t = d + w$, $w \geqslant 1$. Let $T^<$ be a colored Yamanouchi tableau of shape ν, content (m^t), and total color d. Then for every letter $\bar{s}$ in $T^<$ (and thus in T^b), we have $s \geqslant \ell(\eta) + w$.*

Proof. By remark 4.2, if $\bar{s}$ in $T^<$, then $s \geqslant t - \eta_1 + 1$. For any partition $\eta \vdash d$, we have $\eta_1 + \ell(\eta) \leqslant d + 1$. Thus, $s \geqslant t + \ell(\eta) - d = \ell(\eta) + w$. $\square$

Corollary 4.4. *If $T^<$ is as in proposition 4.3 and $j = \max\{s | \bar{s} \text{ is not in } T^b\}$, then $j \geqslant \ell(\eta) + w - 1$.*

Next, we describe some properties of the fillings of Yamanouchi tableaux.

Proposition 4.5. *Let T be a colored Yamanouchi tableau. Then for all $i, j \geqslant 1$, $T^{\prec}_{i,j} \preceq i$. Moreover, if $T^{\prec}_{i,j}$ is unbarred, we have $i - \eta'_j \preceq T^{\prec}_{i,j}$.*

Proof. Let $p = T^{\prec}_{i,j}$. If p is barred, then $p \preceq i$ by the definition of the small bar order. If p is unbarred, then p is in $T^{\prec}/T^b$. Then $p \leqslant i$, and therefore $p \preceq i$, by the lattice permutation condition. By the semistandard condition, the unbarred letters in each column are strictly increasing. Therefore, $i - \eta'_j \preceq T^{\prec}_{i,j}$. □

We denote by $c^{\varnothing}(T)$ the content of the unbarred letters in a colored tableau T. We simply write $c^{\varnothing}$ if it is obvious which tableau we refer to. Then, $c_i^{\varnothing}$ denotes the number of unbarred i in T.

Proposition 4.6. *Consider a colored Yamanouchi tableau $T^{\prec}$ of shape ν. Let $\ell(\eta) + 1 \leqslant p < \ell(\nu)$. If $c_q^{\varnothing} = c_{q+1}^{\varnothing}$ and $T^{\prec}_{p,1} = q$, then $T^{\prec}_{p+1,1} = q + 1$.*

Proof. By the Yamanouchi condition, we must have a $q + 1$ in some row $p' > p$. By the semistandard condition, $T^{\prec}_{p+1,1} \geqslant q + 1$. If $T^{\prec}_{p+1,1} > q + 1$, by the semistandard condition, there cannot be any $q + 1$ in row $p + 1$, or any row below it. Therefore, $T^{\prec}_{p+1,1} = q + 1$. □

Corollary 4.7. *If $c_q^{\varnothing} = c_{q+1}^{\varnothing} = \cdots = c_{q+u}^{\varnothing}$ for some $u > 0$ and $T^{\prec}_{p,1} = q$, then $T^{\prec}_{p+u,1} = q + u$.*

The next proposition establishes bounds for the lengths of rows in Yamanouchi tableaux of shape ν and of content (m^t). Again, the fact that the tableau is color raisable is irrelevant.

Proposition 4.8. *Consider a colored Yamanouchi tableau T of shape ν with content (m^t). Then, for $1 \leqslant i \leqslant \ell(\nu)$, we have $c_i^{\varnothing} \leqslant \nu_i \leqslant m + \eta_i$.*

Proof. In $T^{\prec}$, the unbarred letter i appears $c_i^{\varnothing}$ times. By proposition 4.5, each unbarred i must be in or below row i, and by the semistandard condition each unbarred i is in its own column. Therefore, $c_i^{\varnothing} \leqslant \nu_i$.

Let S be the tableau consisting of the first $i - 1$ rows of $T^{\prec}/T^b$, and let $f = (f_1, f_2, \ldots, f_{i-1})$ be the content of S. (Note that some of the last entries in f could be 0.) Since there are m of the letter 1 (barred or unbarred), row i in $T^{\prec}/T^b$ may contain the letter 1 at most $m - f_1$ times. By the lattice permutation condition, for each $j = 2, 3, \ldots, i - 1$, row i in $T^{\prec}$ contains the letter j at most $f_{j-1} - f_j$ times. Recall that $f_i = 0$. No letter larger than i can appear in row i of $T^{\prec}/T^b$. Thus, we have

$$\nu_i - \eta_i \leqslant (m - f_1) + (f_1 - f_2) + (f_2 - f_3) + \cdots + (f_{i-1} - f_i) = m.$$

Therefore, $c_i^{\varnothing} \leqslant \nu_i \leqslant m + \eta_i$. □

The next theorem, while a simple consequence of the previous statements, is crucial for proving the superstability of Kronecker coefficients.

Theorem 4.9. *Let $t = d + w$ and assume that $w \geqslant 2$. Let T be a colored Yamanouchi tableau of shape ν, content (m^t), and total color d. Then, for $1 \leqslant p \leqslant w - 1$, we have $\nu_{\ell(\eta)+p} = m$.*

Proof. Let $j = \max\{s | \bar{s} \text{ is not in } T^b\}$. By corollary 4.4, $j \geqslant \ell(\eta) + w - 1$. Thus, if $1 \leqslant p \leqslant w - 1$, then $j \geqslant \ell(\eta) + p$. Then, in $T^{\prec}$, all m letters $\ell(\eta) + p$ are unbarred.

Clearly, $\eta_{\ell(\eta)+p} = 0$. Therefore, by proposition 4.8, we have $m \leqslant \nu_{\ell(\eta)+p} \leqslant m$. It follows that $\nu_{\ell(\eta)+p} = m$. □

It follows easily from the proof of proposition 4.8 that if $\nu_{\ell(\eta)+1} = m$, the first $\ell(\eta)$ rows of $T^{\prec}$ completely determine the filling of row $\ell(\eta) + 1$.

Corollary 4.10. *Let T be a colored Yamanouchi tableau of shape ν, content (m^t), and total color d. Suppose $t = d + w$, $w \geqslant 1$. If $\nu_{\ell(\eta)+1} = m$, then $\zeta_i = \xi_{i-1} - \xi_i$ for all $1 \leqslant i \leqslant \ell(\eta) + 1$. Moreover, if $w \geqslant 3$, then ξ also determines the filling of rows $\ell(\eta) + 2, \ldots, \ell(\eta) + w - 1$ in $T^{\prec}$.*

Proof. Since $\nu_{\ell(\eta)+1} = m$, by the proof of proposition 4.8, for each $1 \leqslant i \leqslant \ell(\eta) + 1$, the letter i appears in row $\ell(\eta) + 1$ of $T^{\prec}$ exactly $\xi_{i-1} - \xi_i$ times; and these are precisely the letters in row $\ell(\eta) + 1$. By the same argument, for $2 \leqslant k \leqslant w - 1$, each box in row $\ell(\eta) + k$ is filled with the letter obtained by adding 1 to the letter directly above it. □

As shown in theorem 4.9, if $t = d + w$, $w \geqslant 2$ and $g((m^t), \mu(d), \nu) > 0$, then $\nu_{\ell(\eta)+1} = m$. Since our goal is to show that the superstability phenomenon occurs starting with $t = d + 1$, we examine this case separately. However, since we seek a bijection between tableaux with $t = d + 1$ and tableaux with $t = d + 2$, we prove some of the following statements for $t = d + w$, $w \geqslant 1$.

Next, we show that if $t = d + 1$, row $\ell(\eta) + 1$ has length m or $m - 1$, unless η is a column. In particular, if η is not a hook, we will show that $\nu_{\ell(\eta)+1} = m$ and corollary 4.10 applies.

The next proposition establishes a criterion for η to be a hook.

Proposition 4.11. *Let $t = d + w$, $w \geqslant 1$. Let T be a colored Yamanouchi tableau of shape ν, content (m^t), and total color d. Let $j = \max\{s | \bar{s}$ is not in $T^b\}$. Then $j = \ell(\eta) + w - 1$ if and only if η is a hook.*

Proof. Recall that, by remark 4.2, the smallest s such that $\bar{s}$ is in T^b equals $t - \eta_1 + 1$. Then

$$j + 1 = d + w - \eta_1 + 1.$$

Therefore,

$$\ell(\eta) + w - 1 = j \Longleftrightarrow \ell(\eta) = j + 1 - w \Longleftrightarrow$$
$$\ell(\eta) = d - \eta_1 + 1 \Longleftrightarrow \eta_1 + \ell(\eta) - 1 = d \Longleftrightarrow \eta \vdash d \text{ is a hook.}$$

□

Proposition 4.12. *Let T be a colored Yamanouchi tableau of shape ν, with content (m^{d+1}). If η is not a hook, then $\nu_{\ell(\eta)+1} = m$.*

Proof. By corollary 4.4 and proposition 4.11, since η is not a hook, $\ell(\eta) < j$. Thus, $\ell(\eta) + 1 \leqslant j$ and $c^{\emptyset}_{\ell(\eta)+1} = m$. Proposition 4.8 implies that $m \leqslant \nu_{\ell(\eta)+1} \leqslant m$. Therefore, $\nu_{\ell(\eta)+1} = m$. □

We now examine the case when η is a hook by considering two cases: η is a column, i.e., $\eta = (1^d)$, and η is a hook but not a column. We will begin by considering η to be a non-column hook, and we will prove several propositions related to the shape of ν and filling of T.

Note that if $t = d + w$, $w \geqslant 1$, and η is a non-column hook, then, by proposition 4.11 and the proof of theorem 4.1, the first row of T^b is filled with the barred letters $\overline{\ell(\eta) + w}$, $\overline{\ell(\eta) + w + 1}$, …, $\overline{d + w}$ and each of the remaining rows of T^b has one box filled with a $\overline{d + w}$. Thus, every colored Yamanouchi tableau T of shape ν, with content (m^t), and with non-column hook η has

$$c^{\varnothing} = \left(m^{\ell(\eta)+w-1}, (m-1)^{d-\ell(\eta)}, m - \ell(\eta)\right). \tag{2}$$

If we set $\ell(\eta) = d$ in (2) we obtain $c^{\varnothing}$ for the case when $\eta = (1^d)$ is a column.

Proposition 4.13. *Let T be a colored Yamanouchi tableau of shape ν, with content (m^{d+1}), and with non-column hook η. Then, $m - 1 \leqslant \nu_{\ell(\eta)+1} \leqslant m$.*

Proof. Since η is not a column, $d - \ell(\eta) > 0$, and therefore $c^{\varnothing}_{\ell(\eta)+1} = m - 1$. Moreover, $\eta_{\ell(\eta)+1} = 0$. Thus, by proposition 4.8, $m - 1 \leqslant \nu_{\ell(\eta)+1} \leqslant m$. □

Proposition 4.14. *Let T be a colored Yamanouchi tableau of shape ν, with content (m^{d+w}), $w \geqslant 1$, and non-column hook η. If $\nu_{\ell(\eta)+w} < m$, then there is an unbarred $\ell(\eta) + w$ in the first column of $T^{\prec}$.*

Proof. In $T^{\prec}$, all $m - 1$ of the unbarred letters $\ell(\eta) + w$ are in or below row $\ell(\eta) + w$, and no two are in the same column. Thus, if there is no unbarred $\ell(\eta) + w$ in the first column, then $\nu_{\ell(\eta)+w} \geqslant (m - 1) + 1 = m$. □

Proposition 4.15. *Let T be a colored Yamanouchi tableau of shape ν, with content (m^{d+1}), and non-column hook η. Suppose that $\nu_{\ell(\eta)+1} = m - 1$. For all $\ell(\eta) + 1 \leqslant p < \ell(\nu)$, if $T^{\prec}_{p,1} = q$ and $T^{\prec}_{p+1,1}$ exists, then $T^{\prec}_{p+1,1} = q + 1$.*

Proof. By proposition 4.6 the statement is true if $c^{\varnothing}_q = c^{\varnothing}_{q+1}$. Suppose that $c^{\varnothing}_q \neq c^{\varnothing}_{q+1}$. From (2) it follows that either $q = \ell(\eta)$ (and $c^{\varnothing}_q = m$, $c^{\varnothing}_{q+1} = m - 1$) or $q = d$ (and $c^{\varnothing}_q = m - 1$, $c^{\varnothing}_{q+1} = m - \ell(\eta)$).

(i) $q = \ell(\eta)$: Since $\nu_{\ell(\eta)+1} = m - 1$, by proposition 4.14 there is an $\ell(\eta) + 1$ in the first column. By the semistandard condition, $\ell(\eta) + 1$ must be directly below $\ell(\eta)$ in the first column.

(ii) $q = d$: By the semistandard condition, $T^{\prec}_{p+1,1} = d + 1$, the only unbarred letter larger than d. □

Next, we examine the content of the first $\ell(\eta) + 1$ rows in a tableau $T^{\prec}$.

Given a Yamanouchi colored tableau T with non-column hook η, let $k = T^{\prec}_{\ell(\eta)+1,1}$ and let y be the number of letters $\ell(\eta)$ in row $\ell(\eta)$ of $T^{\prec}$. By proposition 4.5, $k \leqslant \ell(\eta) + 1$. As before, we denote by ξ be the content of T_1. Let $\tilde{\zeta}$ be the content of the $m - 1$ boxes of row $\ell(\eta) + 1$ in $T^{\prec}$. Note that row $\ell(\eta) + 1$ might have length m, but we are only interested in the content of the first $m - 1$ boxes.

Proposition 4.16. *Let T be a colored Yamanouchi tableau of shape ν, with content (m^{d+w}), $w \geqslant 1$, and non-column hook η.*

(i) If $k = \ell(\eta) + 1$, then $\xi = (m^{\ell(\eta)})$ and $\tilde{\zeta} = (0^{\ell(\eta)}, m - 1)$.
(ii) If $k = \ell(\eta)$, then $\xi = (m^{\ell(\eta)-1}, y)$ and $\tilde{\zeta} = (0^{\ell(\eta)-1}, m - y, y - 1)$.
(iii) If $k < \ell(\eta)$, then $\xi = (m^{k-1}, (m-1)^{\ell(\eta)-k}, y)$ and $\tilde{\zeta} = (0^{k-1}, 1, 0^{\ell(\eta)-k-1}, m - y - 1, y - 1)$.

Proof. First, we make some observations. The only possible (unbarred) letters in T_1 are $1, 2, \ldots, \ell(\eta)$. By (2), each $1 \leqslant i \leqslant \ell(\eta)$ in T is unbarred. Consider a letter $i < k$. Since $k = T^{\prec}_{\ell(\eta)+1,1}$, no i is in or below row $\ell(\eta) + 1$. Thus, all m unbarred i are in the first $\ell(\eta)$ rows. There are $k - 1$ such letters. Then, each of the first $k - 1$ parts of ξ equals m. Since η is a non-column hook, by proposition 4.5, $\ell(\eta) \leqslant T_{\ell(\eta)+1,2} \leqslant \ell(\eta) + 1$. Thus, row $\ell(\eta) + 1$ in $T^{\prec}$ is filled with only the letters k, $\ell(\eta)$, and $\ell(\eta) + 1$.

(i) Suppose $k = \ell(\eta) + 1$. Then $k - 1 = \ell(\eta)$ and $\xi = (m^{\ell(\eta)})$. Moreover, by the semistandard condition, row $\ell(\eta) + 1$ is filled with $\ell(\eta) + 1$ and $\tilde{\zeta} = (0^{\ell(\eta)}, m - 1)$.

(ii) Suppose $k = \ell(\eta)$. Since $k - 1 = \ell(\eta) - 1$, we have $\xi = (m^{\ell(\eta)-1}, y)$. If $k = \ell(\eta)$, then there is no letter $\ell(\eta)$ in any row below row $\ell(\eta) + 1$, and thus $T^{\prec}_{\ell(\eta)+1,i} = \ell(\eta)$ for $1 \leqslant i \leqslant m - y$ and $T^{\prec}_{\ell(\eta)+1,i} = \ell(\eta) + 1$ for $m - y + 1 \leqslant i \leqslant m - 1$. Thus, $\tilde{\zeta} = (0^{\ell(\eta)-1}, m - y, y - 1)$.

(iii) Suppose $k < \ell(\eta)$. By corollary 4.7, each letter i, $k \leqslant i < \ell(\eta)$, appears in the first column of $T^{\prec}$. Since η is a non-column hook, by proposition 4.5 $\ell(\eta) \leqslant T_{\ell(\eta)+1,2}$. Thus, each $k \leqslant i < \ell(\eta)$ appears in $T^{\prec}$ strictly below row $\ell(\eta)$ exactly once, and $\xi = (m^{k-1}, (m-1)^{\ell(\eta)-k}, y)$. Similarly, $\ell(\eta)$ appears in $T^{\prec}$ strictly below row $\ell(\eta) + 1$ exactly once. Then, $T^{\prec}_{\ell(\eta)+1,i} = \ell(\eta)$ for $2 \leqslant i \leqslant m - y$ and $T^{\prec}_{\ell(\eta)+1,i} = \ell(\eta) + 1$ for $m - y + 1 \leqslant i \leqslant m - 1$. Therefore, $\tilde{\zeta} = (0^{k-1}, 1, 0^{\ell(\eta)-k-1}, m - y - 1, y - 1)$. □

The next corollary, which follows directly from proposition 4.16, gives an analog of corollary 4.10 in the case when $\nu_{\ell(\eta)+1} = m - 1$.

Corollary 4.17. *Let T be a colored Yamanouchi tableau T of shape ν, with content (m^{d+1}), and non-column hook η. Suppose that $\nu_{\ell(\eta)+1} = m - 1$. Set $\xi_0 = m$. Then, $\tilde{\zeta}_i = \zeta_i = \xi_{i-1} - \xi_i$ for $1 \leqslant i \leqslant \ell(\eta)$, and $\tilde{\zeta}_{\ell(\eta)+1} + 1 = \xi_{\ell(\eta)} - \xi_{\ell(\eta)+1}$.*

We need to establish a lower bound for the length of row $\ell(\eta)$ of a tableau counted by Blasiak's rule. If η is not a column, this is accomplished in the next corollary. In propositions 4.21 and 4.22 we show that the bound also holds if η is a column.

Corollary 4.18. *Let T be a colored Yamanouchi tableau of shape ν, with content (m^{d+w}), $w \geqslant 1$, and non-column η. Then, $\nu_{\ell(\eta)} \geqslant m$.*

Proof. If $w \geqslant 2$, this follows from theorem 4.9. If $w = 1$ and η is not a hook, this follows from proposition 4.12. If η is a non-column hook, by proposition 4.16

$$\xi_{\ell(\eta)} + \tilde{\zeta}_{\ell(\eta)} = \begin{cases} m & \text{if } \ell(\eta) \leqslant k \leqslant \ell(\eta) + 1 \\ m - 1 & \text{if } k < \ell(\eta) \end{cases}$$

and the statement follows. □

$T^{\prec}_{1,1}$	$T^{\prec}_{1,2}$	$T^{\prec}_{1,3}$	$\dots$	$T^{\prec}_{1,k}$	$\dots$
$T^{\prec}_{2,1}$	a_2	$\dots$			
$T^{\prec}_{3,1}$	a_3	$\dots$			
$\dots$	$\dots$	$\dots$			
$T^{\prec}_{\ell(\eta),1}$	$a_{\ell(\eta)}$	$\dots$			
b_1	c_1	$\dots$			
b_2	c_2	$\dots$			
$\dots$	$\dots$	$\dots$			
b_r	c_r	$\dots$			

Figure 6. Labeling scheme for tableaux with non-column hook η.

We now consider the color raisable property for tableaux with non-column hook η. We use the following labeling scheme in a tableau $T^{\prec}$. Let $a_i = T^{\prec}_{i,2}$ for $2 \leqslant i \leqslant \ell(\eta)$, $b_i = T^{\prec}_{\ell(\eta)+i,1}$, and $c_i = T^{\prec}_{\ell(\eta)+i,2}$. We illustrate this in figure 6. Note that not all columns are necessarily of equal length. Boxes to the right of the first column should be viewed as missing from the figure as needed.

If $\eta = (d)$, $d \geqslant 2$, i.e., η is a row, we set $a_1 = 0$. Then, proposition 4.19 below still holds, and proposition 4.20 does not apply. One can also check directly that if $\eta = (d)$ and $d \geqslant 2$, then $T^{<}$ is color raisable.

Proposition 4.19. *Let T be a colored Yamanouchi tableau of shape ν, with content (m^{d+w}), $w \geqslant 1$, and non-column hook η. Suppose that $\nu_{\ell(\eta)+w} = m - 1$. If $a_{\ell(\eta)} < b_1$, then $T^{<}$ is color raisable.*

Proof. If $a_{\ell(\eta)} < b_1$, when converting to the natural order, $T^{\prec}_{\ell(\eta),1}$ moves to the right. Next, $T^{\prec}_{\ell(\eta)-1,1}$ has a neighbor $a_{\ell(\eta)}$ below and a neighbor $a_{\ell(\eta)-1}$ to its right. Since $a_{\ell(\eta)-1} < a_{\ell(\eta)}$, $T^{\prec}_{\ell(\eta)-1,1}$ moves to the right. Similarly, each $T^{\prec}_{i,1}$, $i \geqslant 2$, moves to the right. We have $T^{\prec}_{1,1} = \overline{\ell(\eta) + w}$. Since $\nu_{\ell(\eta)+w} \leqslant m - 1$, by proposition 4.14, there is an unbarred $\ell(\eta) + w$ in the first column of $T^{\prec}$. Thus, T is color raisable. □

Proposition 4.20. *Let T be a colored Yamanouchi tableau of shape ν, with content (m^{d+w}), $w \geqslant 1$, and non-column hook η. Suppose that $b_k = b_1 + k - 1$ for all $1 \leqslant k \leqslant \ell(\nu) - \ell(\eta)$. If $b_1 \leqslant a_{\ell(\eta)}$, then $T^{<}$ is color raisable if and only if $T^{\prec}_{\ell(\nu),1} = d + w$.*

$\cdots$	$\cdots$
b_r	c_{r-1}
$T^{\prec}_{\ell(\eta),1}$	c_r

Figure 7. Shifting of $T^{\prec}_{\ell(\eta),1}$ if $b_r = T^{\prec}_{\ell(\eta),1} \neq d + w$.

Proof. By the semistandard condition, if c_k exists, then $a_{\ell(\eta)} + k \leqslant c_k$. Therefore,

$$b_k = b_1 + k - 1 \leqslant a_{\ell(\eta)} + k - 1 \leqslant c_{k-1}.$$

Then, when converting to the natural order, $T^{\prec}_{\ell(\eta),1} = \overline{d+w}$ always moves down (as long as $b_k < d + w$). Thus, $T^{\prec}_{\ell(\eta),1}$ replaces b_{r-1}. If $b_r = T^{\prec}_{\ell(\nu),1} = d + w$, $T^{\prec}_{\ell(\eta),1}$ can no longer move, and $T^{<}$ is color raisable. If $b_r = T^{\prec}_{\ell(\nu),1} \neq d + w$, then $T^{\prec}_{\ell(\eta),1}$ will move down again, resulting in the situation shown in figure 7.

By the Yamanouchi condition, $b_r = d + w - 1$ or $b_r = d + w$. Since $b_r \neq d + w$, we must have $b_r = d + w - 1$. Then, either c_{r-1} does not exist, $c_{r-1} = d + w - 1$, or $c_{r-1} = d + w$. If c_{r-1} does not exist or $c_{r-1} = d + w$, then c_r does not exist. If $c_{r-1} = d + w - 1$, then $c_r = d + w$ or c_r does not exist. In either case, $T^{\prec}_{\ell(\eta),1}$ does not move to the right and $T^{<}$ is not color raisable. $\square$

Finally, we consider the case when $t = d + w$, $w \geqslant 1$, and $\eta = (1^d)$ is a column. This case is simple enough to completely describe every possible filling of $T^{\prec}$, and doing so is useful in deciding whether $T^{\prec}$ is color raisable. In $T^{\prec}$, every box of η is filled with $\overline{d+w}$. Thus, if η is a column, we have $d \leqslant m$. Note that if $\ell(\nu) = \ell(\eta)$, then necessarily $w = 1$ and $m = d$. The only possible tableau $T^{\prec}$ in this case has shape $((d+1)^d)$, and for each $1 \leqslant i \leqslant d$, row i is filled with a $\overline{d+1}$ followed by m letters i. The corresponding $T^{<}$ is obtained by moving each barred letter to the end of its row. Thus $T^{<}$ is color raisable. For the remainder of the discussion we assume that $\ell(\nu) > \ell(\eta)$. As before, let $k = T^{\prec}_{\ell(\eta)+1,1}$.

We consider first the case $m = d$.

Proposition 4.21. *Let T a colored Yamanouchi tableau of shape ν, with content (d^{d+w}), $w \geqslant 1$, column $\eta = (1^d)$, and $\ell(\nu) > \ell(\eta)$. Then*

$$\nu = \left((d+1)^{k-1}, d^{d-k+w}, 1^{d-k+1}\right). \tag{3}$$

Conversely, for each $1 \leqslant k \leqslant \min\{d + w - 1, d + 1\}$, there is exactly one colored Yamanouchi tableau T of shape ν as in (3), with content (d^{d+w}), total color d, and $\eta = (1^d)$. Moreover, $T^{<}$ is color raisable if and only if $k = d + 1$.

Proof. Since $c_i^{\varnothing} = m$ for all $1 \leqslant i \leqslant d + w - 1$ and $c_{d+w}^{\varnothing} = 0$, one can easily see that k completely determines the shape ν and the filling of $T^{\prec}$ (which in turn determines the filling of $T^{<}$). By corollary 4.7, the first column of $T^{\prec}/T^{b}$ is filled, from top to bottom, with $k, k+1, \ldots, d+w-1$. By proposition 4.5, $T^{\prec}_{i,j} = i$ for all $1 \leqslant i \leqslant d + w - 1$ and all $j \geqslant 2$. If $k \leqslant d$, when converting to the natural order, the bottom $\overline{d+w}$ moves down at every step and $T^{<}$ is not color raisable. If $k = d + 1$, each $\overline{d+w}$ moves to the right and $T^{<}$ is color raisable. $\square$

Thus, if $m = d$ and $t = d + 1$, the only color raisable tableau has shape $((d + 1)^d)$ with $T^{\prec}$ filled as explained in the discussion preceding proposition 4.21. Similarly, if $m = d$ and $t = d + w$, the only color raisable tableau has shape $((d + 1)^d, d^{w-1})$ with $T^{\prec}$ filled as in the proof of proposition 4.21.

Now we consider the case $m > d$.

Proposition 4.22. *Let T be a colored Yamanouchi semistandard tableau of shape* ν, *with content* (m^{d+w}), $m > d$, $w \geqslant 1$, *and with column* $\eta = (1^d)$. *Then*

$$\nu = \left((m + 1)^{k-1}, m^{d-k+w}, m - d, 1^{d-k+1}\right) \tag{4}$$

or

$$\nu = \left((m + 1)^{k-1}, m^{d-k+w}, 1 + m - d, 1^{d-k}\right). \tag{5}$$

(If $k = d + 1$, *then (5) does not occur.)*

Conversely, for each $1 \leqslant k \leqslant d + 1$, *there is exactly one colored Yamanouchi tableau T of shape* ν *given by (4) or (5), with content* (m^{d+w}), *total color d, and* $\eta = (1^d)$. *Moreover,* $T^{<}$ *is color raisable if and only if* ν *is given by (4).*

Proof. Since $c_i^{\emptyset} = m$ for $1 \leqslant i \leqslant d + w - 1$, the letter k completely determines the placement of letters $1, 2, \ldots, d + w - 1$. Once these letters are placed, the resulting tableau $T^{\prec,\text{top}}$ has shape $\nu^{\text{top}} = ((m + 1)^{k-1}, m^{d-k+w}, 1^{d-k+1})$ with filling as in the proof of proposition 4.21.

If $k = d + 1$, there is no row of length 1 in $T^{\prec,\text{top}}$ and all $m - d$ labels $d + w$ must be placed after the last row of $T^{\prec,\text{top}}$ thus creating a tableau of shape ν as in (4).

If $k \leqslant d$, there are two choices for the placement of the remaining $m - d$ letters $d + w$:

(i) one label $d + w$ is placed at the end of the first column and the remaining $m - d - 1$ letters $d + w$ are placed in row $d + w$, creating a tableau of shape ν as in (4), or

(ii) all remaining $m - d$ letters $d + w$ are placed in row $d + w$, creating a tableau of shape ν as in (5).

Suppose ν is as in (4). If $k = d + 1$, when converting to the natural order all barred letters move to the right, and $T^{<}$ is color raisable. If $k \leqslant d$, the entry in the SW corner of $T^{\prec}$ is $d + w$, and $T^{<}$ is color raisable.

If ν is as in (5), at every step, the lowest $\overline{d + w}$ moves down. Since the entry in the SW corner of $T^{\prec}$ is $d + w - 1$, $\overline{d + w}$ reaches the SW corner. If $k < d$, the last row of ν has length 1. If $k = d$, each of the $m - d$ boxes in the last row of $T^{\prec}$, to the right of $d + w - 1$, is filled with $d + w$. Thus, $\overline{d + w}$ does not shift from the SW corner to the right and $T^{<}$ is not color raisable. $\square$

Corollary 4.23. *Let T be a colored Yamanouchi semistandard tableau of shape* ν, *with content* (m^{d+w}), $w \geqslant 1$, *and with column* $\eta = (1^d)$. *Then,* $\nu_{\ell(\eta)} = \nu_d \geqslant m$.

5. Stability of the the Kronecker coefficients

In this section we state and prove our main result, a stability property for the Kronecker coefficients in the case when one partition is a hook and the other is a rectangle. Moreover, we give a bound for the size of the partition starting with which the Kronecker coefficients are stable. We also show that once the stability bound is reached, no new Schur function appear in the decomposition of the Kronecker product. We call this property *superstability*. We first introduce some notation.

Fix integers $m \geqslant 2$ and $d, w \geqslant 1$. Let $t = d + w$, and fix a partition $\nu \vdash mt$. Let $A_w^{\nu,\prec}$ be the collection of Yamanouchi colored tableaux $T^\prec$ (for the *small bar order*) of shape ν, content (m^t), and total color d. Given $T^\prec \in A_w^{\nu,\prec}$, we define a tableau $\varphi^\nu(T^\prec)$ by performing the steps below on $T^\prec$. We denote by R the unique SSYT tableau of (row) shape (m) and content ρ, with $\rho_i = \xi_{i-1} - \xi_i$, $1 \leqslant i \leqslant \ell(\eta) + 1$.

(i) Increase each barred letter of T^b by 1.
(ii) Keep T_1 unchanged.
(iii) Insert row R between T_1 and T_2 (i.e., after row $\ell(\eta)$).
(iv) Increase each letter of T_2 by 1.

Let $\nu^{(m)}$ denote the partition obtained from ν by adding a part of length m and rearranging the parts to form a partition. Since $\nu_{\ell(\eta)} \geqslant m$, $\varphi^\nu(T^\prec)$ has shape $\nu^{(m)}$. Clearly, $\varphi^\nu(T^\prec)$ has total color d, and it is straightforward to check that it has content (m^{t+1}). In the next proposition we prove that $\varphi^\nu(T^\prec)$ is Yamanouchi, thus showing that φ^ν defines a map from $A_w^{\nu,\prec}$ to $A_{w+1}^{\nu^{(m)},\prec}$.

First we introduce some notation. Let v be the rrw of T_1, and let $\mathfrak{a}$ be the rrw of $(T^\prec)^{sk}/T_1$. Denote by ${}^+\mathfrak{a}$ (respectively, ${}^-\mathfrak{a}$) the word obtained from $\mathfrak{a}$ by increasing (respectively decreasing) each letter by 1. Let $\mathfrak{b}$ be the rrw of R. Then the concatenated word $v\mathfrak{a}$ is the rrw of $(T^\prec)^{sk}$. The concatenated word $v\mathfrak{b}({}^+\mathfrak{a})$ is the rrw of $(\varphi^\nu(T^\prec))^{sk}$. Given a word u, we denote by $\ell(u)$ the length of u, i.e., the number of letters in u. We also denote by $u^{(i)}$ the number of is in u.

Proposition 5.1. *If $T^\prec \in A_w^{\nu,\prec}$, then $\varphi^\nu(T^\prec) \in A_{w+1}^{\nu^{(m)},\prec}$.*

Proof. Let $T^\prec \in A_w^{\nu,\prec}$. Thus, $T^\prec$ is Yamanouchi, i.e., the word $v\mathfrak{a}$ is a lattice permutation. We need to show that $\varphi^\nu(T^\prec)$ is Yamanouchi, i.e., the word $v\mathfrak{b}({}^+\mathfrak{a})$ is a lattice permutation. Let u be an initial factor of $v\mathfrak{b}({}^+\mathfrak{a})$. We need to show that $u^{(i)} \geqslant u^{(i+1)}$ for all i. If $\ell(u) \leqslant \ell(v\mathfrak{b})$, this is true by the definition of row R and the fact that $v\mathfrak{a}$ is lattice permutation. If $\ell(u) > \ell(v\mathfrak{b})$, let $\tilde{u}$ be the subword of u obtained by deleting $v\mathfrak{b}$ from its beginning. Thus, ${}^-\tilde{u}$ is an initial factor of $\mathfrak{a}$. Then

$$\begin{aligned} u^{(i)} - u^{(i+1)} &= \xi_i + \rho_i + \tilde{u}^{(i)} - \left(\xi_{i+1} + \rho_{i+1} + \tilde{u}^{(i+1)}\right) = \xi_{i-1} - \xi_i + \tilde{u}^{(i)} - \tilde{u}^{(i+1)} \\ &= \xi_{i-1} + ({}^-\tilde{u})^{(i-1)} - \left(\xi_i + ({}^-\tilde{u})^{(i)}\right) = \left(v({}^-\tilde{u})\right)^{(i-1)} - \left(v({}^-\tilde{u})\right)^{(i)}) \geqslant 0. \end{aligned}$$

The last inequality holds because $v({}^-\tilde{u})$ is an initial factor of $v\mathfrak{a}$. □

Recall that, by corollary 4.10, if $w \geqslant 1$ (and thus $w + 1 \geqslant 2$) and $\tilde{T}^\prec$ is a colored Yamanouchi tableau of shape $\tilde{\nu}$, content (m^{d+w+1}), and total color d, then $\tilde{\nu}_{\ell(\eta)+1} = m$ and row $\ell(\eta) + 1$ has content ζ with $\zeta_i = \xi_{i-1} - \xi_i$. If ν is the partition obtained from $\tilde{\nu}$ by removing a part of length m, then $\tilde{\nu} = \nu^{(m)}$. Thus, if $w \geqslant 1$ and $\tilde{T}^\prec \in A_{w+1}^{\nu^{(m)},\prec}$, we can reverse the steps in the definition of $\varphi^\nu(T^\prec)$. In the next theorem we show that the obtained tableau is in $A_w^{\nu,\prec}$.

Theorem 5.2. *The map $\varphi^\nu : A_w^{\nu,\prec} \to A_{w+1}^{\nu^{(m)},\prec}$ is a bijection.*

Proof. Let $\tilde{T}^\prec \in A_{w+1}^{\nu^{(m)},\prec}$. Thus, $\tilde{T}^\prec$ is Yamanouchi. Let $T^\prec$ be the tableau obtained from $\tilde{T}^\prec$ by reversing steps (i)-(iv) above. Then $T^\prec$ has shape ν, content (m^t), and total color d. It remains to show that $T^{\prec}$ is Yamanouchi. With v and $\mathfrak{a}$ as defined above, let u be an initial factor of $v\mathfrak{a}$. We show that $u^{(i)} \geqslant u^{(i+1)}$ for all i. If $\ell(u) \leqslant \ell(v)$, this is true because $\tilde{T}^{\prec}$ is Yamanouchi. If $\ell(u) > \ell(v)$, let $\tilde{u}$ be the subword of u obtained by deleting v from its beginning. Then,

$$u^{(i)} - u^{(i+1)} = \xi_i + \tilde{u}^{(i)} - \left(\xi_{i+1} + \tilde{u}^{(i+1)}\right)$$

$$= \xi_{i+1} + \rho_{i+1} + ({}^{+}\tilde{u})^{(i+1)} - \left(\xi_{i+2} + \rho_{i+2} + ({}^{+}\tilde{u})^{(i+2)}\right)$$

$$= \left(v\mathfrak{b}({}^{+}\tilde{u})\right)^{i+1} - \left(v\mathfrak{b}({}^{+}\tilde{u})\right)^{i+2} \geqslant 0.$$

The last inequality holds because $v\mathfrak{b}({}^{+}\tilde{u})$ is an initial factor of the rrw of $\tilde{T}^{\prec}$. □

We let

$$A_w^{\prec} = \bigcup_{\nu} A_w^{\nu,\prec}$$

and define a map $\varphi : A_w^{\prec} \to A_{w+1}^{\prec}$ as follows. For $T^{\prec} \in A_w^{\prec}$, with shape ν, we set $\varphi(T^{\prec}) = \varphi^{\nu}(T^{\prec})$. Thus, we have proved the following theorem.

Theorem 5.3. *If $w \geqslant 1$, the map $\varphi : A_w^{\prec} \to A_{w+1}^{\prec}$ is a bijection.*

For the rest of the article, if $T^{<}$ is a colored tableau in the natural order, by $(T^{<})^{\prec}$ we mean the tableau obtained when converting $T^{<}$ to the small bar order. Similarly, if $T^{\prec}$ is a colored tableau in the small bar order, by $(T^{\prec})^{<}$ we mean the tableau obtained when converting $T^{\prec}$ to the natural order.

Again, fix integers $m \geqslant 2$, and d, $w \geqslant 1$. Let $t = d + w$, and fix a partition $\nu \vdash mt$. We denote by $B_w^{\nu,<}$ the collection of color raisable Yamanouchi colored tableaux $T^{<}$ (for the *natural order*) of shape ν, content (m^t), and total color d. Let

$$B_w^{<} = \bigcup_{\nu} B_w^{\nu,<}.$$

Given $T^{<} \in B_w^{\nu,<}$, we have $(T^{<})^{\prec} \in A_w^{\nu,\prec}$. We define $\psi^{\nu}(T^{<}) = (\varphi((T^{<})^{\prec}))^{<}$. Thus, $T^{<}$ is converted to the small bar order, mapped by φ to $A_{w+1}^{\prec}$, and then converted back to the natural order. In the next theorem, we show that a colored Yamanouchi tableau $T^{<}$ is color raisable if and only if $\psi^{\nu}(T^{<})$ is color raisable. Thus, we show that ψ^{ν}: $B_w^{\nu,<} \to B_{w+1}^{\nu^{(m)},<}$ is a bijection.

Theorem 5.4. *If $w \geqslant 1$, the map $\psi^{\nu} : B_w^{\nu,<} \to B_{w+1}^{\nu^{(m)},<}$ is a bijection.*

Proof. We show that, given a tableau $T^{\prec} \in A_w^{\prec}$, the SW corner in $(T^{\prec})^{<}$ is barred if and only if the SW corner of $(\varphi(T^{\prec}))^{<}$ is barred. Since, by theorem 5.3, φ is a bijection, this will prove the theorem. Note that η is always the same in $T^{\prec}$ and $\varphi(T^{\prec})$. Let $T^{\prec} \in A_w^{\nu,\prec}$. We have three cases.

(i) $w \geqslant 2$, or $w = 1$ and η non-hook. Then, $\nu_{\ell(\eta)+1} = m$. We show that, when converting both $T^{\prec}$ and $\varphi(T^{\prec})$ to the natural order, all Jeu-de-Taquin moves are essentially the same.

By the proof of corollary 4.10, row R is precisely row $\ell(\eta) + 1$ in $T^{\prec}$. Let R' be row $\ell(\eta) + 2$ in $\varphi(T^{\prec})$. Thus, R' is precisely row R with each entry increased by 1. Let T_1' be the tableau consisting of the first $\ell(\eta) + 1$ rows of $T^{\prec}/T^{b}$, and T_2' be the tableau consisting of the remaining rows of $T^{\prec}/T^{b}$. Then, replacing T_1, T_2, R by T_1', T_2', R' in the steps giving $\varphi(T^{\prec})$ does not change the result.

The smallest letter in T^b (respectively $(\varphi(T^{\prec}))^b$) is at least $\overline{\ell(\eta)+2}$ (respectively $\overline{\ell(\eta)+3}$), and the largest letter in T_1' is at most $\ell(\eta)+1$. Thus, in the natural order, all the unbarred letters in T_1' are smaller than all of the barred letters (in both $T^{\prec}$ and $\varphi(T^{\prec})$.) By propositions 3.1 and 4.5, no unbarred letters greater than $\ell(\eta)+1$ will ever be shifted into T_1'. Therefore, the Jeu-de-Taquin moves that stay in T_1', i.e., in the first $\ell(\eta)+1$ rows, are exactly the same in $T^{\prec}$ and $\varphi(T^{\prec})$.

We start by performing Jeu-de-Taquin moves on all letters $\bar{b}=\overline{d+w}$. Now, suppose a barred letter $\bar{b}$ has arrived by Jeu-de-Taquin moves in row $\ell(\eta)+1$ of $T^{\prec}$, i.e., the last row of T_1', by bumping a label c. Then $\overline{b+1}$ has arrived in the same place in $\varphi(T^{\prec})$. By proposition 3.1, letter $\ell(\eta)+1$ cannot be shifted up to row $\ell(\eta)$ and therefore $c<\ell(\eta)+1$. Then, by the semistandard, lattice permutation, and equal content conditions, the box directly below it must exist and contain $c+1$. Figures 8 and 9 illustrate this situation in $T^{\prec}$ and $\varphi(T^{\prec})$ respectively. In figure 8, the row $\ell(\eta)+1$ is between the lines. In figure 9, viewed from top to bottom, the rows between the lines are rows $\ell(\eta)+1$ and R' respectively. In order for $\bar{b}$ to move down into row $\ell(\eta)+1$ of $T^{\prec}$, we must have $c\leqslant a$. Then, if d exists, we have $c<d$ (and therefore, $c+1\leqslant d$). In each figure, (a) shows the configuration before the shift of $\bar{b}$, respectively $\overline{b+1}$, into row $\ell(\eta)+1$, and (b) shows the shift of $\bar{b}$, respectively $\overline{b+1}$, into row $\ell(\eta)+1$. Figure 9(c) is an additional move in $\varphi(T^{\prec})$, forced by the fact that in $\varphi(T^{\prec})$ the box below c always exists and it is filled with $c+1$. We omitted figure 8(c) to emphasize the additional move in $\varphi(T^{\prec})$. In each figure, (d) shows the next (forced) Jeu-de-Taquin move.

To summarize, in $\varphi(T^{\prec})$, Jeu-de-Taquin moves of $\overline{b+1}$ in row $\ell(\eta)+1$ always go *down* to row R', as in figure 9(c), which is an additional move in $\varphi(T^{\prec})$ versus the moves in $T^{\prec}$. Now, in $\varphi(T^{\prec})$, the part of the tableau below the last line is precisely the tableau T_2' in $T^{\prec}$ with each letter increased by 1, and therefore the remaining moves in $T^{\prec}$ and $\varphi(T^{\prec})$ will be exactly the same.

Therefore, when performing the Jeu-de-Taquin moves to shift $\overline{d+w}$ the moves are essentially the same in $T^{\prec}$ and $\varphi(T^{\prec})$, and we have no barred letter in row $\ell(\eta)+1$. After shifting all $\overline{d+w}$, we remove all boxes containing letters $d+w$ or $\overline{d+w}$ from both tableau. By proposition 3.1, each new tableau is Yamanouchi, and by the rules of Jeu-de-Taquin moves they are both semistandard for the small bar order. From $T^{\prec}$ we obtain a tableau of shape μ and content (m^{d+w-1}), with $\mu_{\ell(\eta)+1}=m$. From $\varphi(T^{\prec})$ we obtain a tableau of shape $\mu^{(m)}$ and content (m^{d+w}). We repeat the converting process recursively.

Thus, the SW corner is either barred or unbarred in both $(T^{\prec})^{<}$ and $(\varphi(T^{\prec}))^{<}$.

(ii) $w=1$ and η is a non-column hook. Notice that when converting from the small bar order to the natural order, barred letters are not bumped by barred letters. Suppose first that $\nu_{\ell(\eta)+1}=m$. In this case, by the same argument as in case (i), the Jeu-de-Taquin moves for all barred letters greater than $\overline{\ell(\eta)+1}$ are essentially the same in $T^{\prec}$ and $\varphi(T^{\prec})$. Once all these barred letters have been shifted, if the SW corner is barred, the shifting of $\overline{\ell(\eta)+1}$ in $T^{\prec}$ (respectively of $\overline{\ell(\eta)+2}$ in $\varphi(T^{\prec})$) will leave it barred. If the SW corner is unbarred and $\nu_{\ell(\eta)+2}\neq 0$, then the shifting of $\overline{\ell(\eta)+1}$ in $T^{\prec}$ (respectively of $\overline{\ell(\eta)+2}$ in $\varphi(T^{\prec})$) will leave it unbarred (as the label in the SW corner of $T^{\prec}$ is at least $\ell(\eta)+1$.) If the SW corner is unbarred and $\nu_{\ell(\eta)+2}=0$, we must have $m=2$ and $d=3$. One can use Maple to verify that superstability holds in this case.

Now suppose $\nu_{\ell(\eta)+1}=m-1$. If $\eta=(d)$, both $T^{\prec}$ and $\varphi(T^{\prec})$ are color raisable. If η is not a row, we use the notation of figure 6 in $T^{\prec}$. In $T^{\prec}$, by proposition 4.15, $b_k=b_1+k-1$ for all $1\leqslant k\leqslant \ell(\nu)-\ell(\eta)$. By construction, the same is true of $\varphi(T^{\prec})$. If $b_1\leqslant a_{\ell(\eta)}$, by proposition 4.20, $T^{\prec}$ is color raisable if and only if $\varphi(T^{\prec})$ is. If $a_{\ell(\eta)}<b_1$, by proposition 4.19, both $T^{\prec}$ and $\varphi(T^{\prec})$ are color raisable.

$$\begin{array}{cc} \bar{b} & a \\ \hline c & d \\ \hline c+1 & f \end{array} \quad \text{(a)} \qquad \begin{array}{cc} c & a \\ \hline \bar{b} & d \\ \hline c+1 & f \end{array} \quad \text{(b)} \qquad \begin{array}{cc} c & a \\ \hline c+1 & d \\ \hline \bar{b} & f \end{array} \quad \text{(d)}$$

Figure 8. Jeu-de-Taquin moves for $\bar{b}$ in $T^{\prec}$.

$$\begin{array}{cc} \overline{b+1} & a \\ \hline c & d \\ \hline c+1 & d+1 \\ \hline c+2 & f+1 \end{array} \quad \text{(a)} \qquad \begin{array}{cc} c & a \\ \hline \overline{b+1} & d \\ \hline c+1 & d+1 \\ \hline c+2 & f+1 \end{array} \quad \text{(b)} \qquad \begin{array}{cc} c & a \\ \hline c+1 & d \\ \hline \overline{b+1} & d+1 \\ \hline c+2 & f+1 \end{array} \quad \text{(c)} \qquad \begin{array}{cc} c & a \\ \hline c+1 & d \\ \hline c+2 & d+1 \\ \hline \overline{b+1} & f+1 \end{array} \quad \text{(d)}$$

Figure 9. Jeu-de-Taquin moves for $\overline{b+1}$ in $\varphi(T^{\prec})$.

(iii) $w = 1$ and η is a column. By propositions 4.21 and 4.22, $T^{\prec}$ is color raisable if and only if $\varphi(T^{\prec})$ is. □

We define a map $\psi: B_w^{<} \to B_{w+1}^{<}$ as follows. For $T^{<} \in B_w^{<}$, with shape ν, we set $\psi(T^{<}) = \psi^{\nu}(T^{<})$.

Theorems 5.3 and 5.4, together with Blasiak's combinatorial rule, lead to our main theorem.

Theorem 5.5 ***Superstability.*** *Fix integers $m \geqslant 1$ and $d \geqslant 0$. Then, whenever $t \geqslant d + 1$, we have*

$$g\Big(\big(m^t\big), \big(n - d, 1^d\big), \nu\Big) = g\Big(\big(m^{t+1}\big), \big(n - d + m, 1^d\big), \nu^{(m)}\Big),$$

where $n = mt$. Moreover, if $g((m^{t+1}), (n - d + m, 1^d), \gamma) > 0$, then there exists $\nu \vdash mt$ such that $\gamma = \nu^{(m)}$.

6. Final remarks

As stated in theorem 5.5, the superstability property for the Kronecker product of a Schur function indexed by a hook partition and one indexed by a rectangular partition proved in this article is much stronger than usual stability properties. Starting with the stability bound, as one increases the size of the partitions by m, no new partitions are introduced. Therefore, if $n = m(d + w)$ and $w \geqslant 1$, one can completely recover the decomposition of the Kronecker product

$$s_{\left(n-d, 1^d\right)} * s_{\left(m^{d+w}\right)}$$

from the Kronecker product

$$s_{\left(m(d+1)-d, 1^d\right)} * s_{\left(m^{d+1}\right)}.$$

We note that the bound $n = tm$, where $t = d + 1$, starting with which superstability holds, is sharp. If $t = d$, superstability fails, as seen in the following example calculated using sage. Let $m = 2$, $d = 3$. Then when $n = md$ (i.e., $w = 0$), we have

$$s_{(2,2,2)} {}^{*} s_{(3,1,1,1)} = s_{(2,2,2)} + s_{(3,1,1,1)} + s_{(3,2,1)} + s_{(4,1,1)} + s_{(4,2)},$$

but when $n = m(d + 1)$ (i.e., $w = 1$), we have

$$\begin{aligned} s_{(2,2,2,2)} {}^{*} s_{(5,1,1,1)} = {} & s_{(2,2,2,2)} + s_{(3,2,1,1,1)} + s_{(3,2,2,1)} + s_{(3,3,1,1)} + s_{(4,1,1,1,1)} \\ & + s_{(4,2,1,1)} + s_{4,2,1,1)} + s_{(4,2,2)} + s_{(4,3,1)} + s_{(5,1,1,1)}. \end{aligned}$$

The bound $n = tm$, where $t = d + 1$, starting with which the regular stability of Kronecker coefficients holds, i.e.,

$$g\Big(\big(m^t\big), \big(n - d, 1^d\big), \nu\Big) = g\Big(\big(m^{t+1}\big), \big(n - d + m, 1^d\big), \nu^{(m)}\Big),$$

is nearly sharp.

If $n = (d - 1)m$, we have verified using sage that

$$g\Big(\big(3^3\big), \big(5, 1^4\big), (5, 2, 1, 1)\Big) = 2$$

while

$$g\Big(\big(3^4\big), \big(8, 1^4\big), (5, 3, 2, 1, 1)\Big) = 3.$$

Thus, the stability property fails in this case. We are uncertain what happens if $w = 0$.

Acknowledgments

The second author would like to thank Dr Dan Kennedy for providing the funds to support this research during the summer of 2014.

References

[BK] Bessenrodt C and Keleshchev A 1999 On Kronecker products of complex representations of the symmetric and alternating groups *Pac. J. Math.* **190** 201–23

[BO-1] Ballantine C and Orellana R 2005 On the Kronecker product $s_{(n-p,p)} * s_\lambda$ *Electron. J. Combin.* **12** R28 (http://www.combinatorics.org/ojs/index.php/eljc/article/view/v12i1r28)

[BO-2] Ballantine C and Orellana R 2005/07 A combinatorial interpretation for the coefficients in the Kronecker product $s_{(n-p,p)} * s_\lambda$ *Seminaire Lotharingien de Combinatoire.* **54A** B54Af (http://eudml.org/doc/230849)

[B] Blasiak J 2012 Kronecker coefficients for one hook shape arXiv:1209.2018

[BMS] Blasiak J, Mulmuley K and Sohoni M 2013 *Geometric Complexity Theory IV: nonstandard quantum group for the Kronecker problem* (*Memoirs of the American Mathematical Society* vol 235) (Providence, RI: American Mathematical Society)

[Br] Brion M 1993 Stable properties of plethysm: on two conjectures of Foulkes *Manuscr. Math.* **80** 347–71

[CM] Clausen M and Meier H 1993 Extreme irreduzible Konstituenten in Tensordarstelhungen symmetrischer Gruppen *Bayreuther Math. Schriften* **45** 1–17

[D] Dvir Y 1993 On the Kronecker product of S_n characters *J. Algebra* **154** 125–40

[LA] Lascoux A 1980 Produit de Kronecker des representations du group symmetrique *Lecture Notes in Mathematics* vol 795 (Berlin: Springer) pp 319–329

[Ma] Macdonal I G 1995 *Symmetric Functions and Hall Polynomials* 2nd edn (Oxford: Oxford University Press)

[Man] Manivel L 2011 On rectangular Kronecker coefficients *J. Algebr. Comb.* **33** 153–62

[M] Murnaghan F D 1938 The analysis of the Kronecker product of irreducible representation of the symmetric group *Am. J. Math.* **60** 761–84

[PP] Pak I and Panova G 2014 Bounds on the Kronecker coefficients arXiv:1406.2988v2

[R] Remmel J 1989 A formula for the Kronecker products of Schur functions of hook shapes *J. Algebra* **120** 100–18

[RW] Remmel J and Whitehead T 1994 On the Kronecker product of Schur functions of two row shapes *Bull. Belg. Math. Soc* **1** 649–83

[Ro] Rosas M H 2001 The Kronecker product of Schur functions indexed by two-row shapes or hook shapes *J. Algebr. Comb.* **14** 153–73

[SS] Sam S V and Snowden A 2015 Proof of Stembridge's conjecture on stability of Kronecker coefficients arXiv:1501.00333v2

[STW] Scharf T, Thibon J-Y and Wybourne B G 1994 Powers of the Vandermonde determinant and the quantum Hall effect *J. Phys. A: Math. Gen* **27** 4211–9

[S] Stanley R P 1999 *Enumerative Combinatorics* vol 2 (Cambridge: Cambridge University Press)

[St] Stembridge J 2014 Generalized stability of Kronecker coefficients (http://math.lsa.umich.edu/~jrs/papers/kron.pdf)

[V1] Vallejo E 1999 Stability of the Kronecker products or irreducible characters of the symmetric group *Electron. J. Comb.* **6** 30

[V2] Vallejo E 2010 Stability of Kronecker coefficients via discrete tomography arXiv:1408.6219

[V3] Vallejo E 2014 A diagrammatic approach to Kronecker squares *J. Comb. Theory* A **127** 243–85